Consistent approach to creation of neutral fermions with anomalous magnetic moment from vacuum by inhomogeneous magnetic field

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Abstract

A consistent approach (based on QFT) to neutral fermion creation (due to their magnetic moments) in strong inhomogeneous magnetic fields is considered. In particular, it is demonstrated that in specific cases, the problem can be technically reduced to the problem of charge particle creation by an electric step. We demonstrate that the latter problem can be solved consistently in the framework of QFT. All the details of such a construction will be described by us in a next publication. Here, however, using some particular results of the general construction and the above mentioned analogy, we calculate neutral particle creation from the vacuum by a linearly growing magnetic field.

1 Introduction

Usually, particle creation from the vacuum by strong electromagnetic fields is associated with creation of charged particles by strong electric-like fields. Acting on virtual charged particles an electric-like field can produce a work and materialize them on the mass shell as real particles. Nevertheless, if a neutral particle has an anomalous magnetic moment, a inhomogeneous magnetic field acting on such a particle, can also change its kinetic energy (produce a work). This mechanism can provide neutral particle creation from the vacuum by strong inhomogeneous magnetic fields. In this respect, one can speak about two candidates (among the known elementary particles), a neutron and a neutrino. It is known that the neutron has negative magnetic moment given by $\mu_n = -1.9130427(5)\mu_N$, where μ_N is the nuclear magneton, $\mu_N = e/2m_N$. It is also possible that neutrinos have magnetic moments, for a review see [1, 2].The recent experimental constraints on the neutrino magnetic moments are in the range $\mu_\nu < 3.2 \times 10^{-11}\mu_B$ [3], where $\mu_B = e/2m_e$ is the Bohr magneton. Astrophysical constraints on the magnetic moment of the Dirac neutrino can be even stronger [4]. Note that in order to satisfy $m_\nu \lesssim 1$eV, the theory argues that a more natural scale for the Dirac neutrino would be $\mu_\nu \lesssim 10^{-14}\mu_B$ [5]. The effect under discussion can be observed in inhomogeneous magnetic fields that have to be very strong in a certain domain. Such fields can exist in the nature. It has been suggested that magnetic fields of order $10^{15} - 10^{16}G$ or stronger, up to $10^{18}G$, can be probably generated during a supernova explosion or in a vicinity of the special group of neutron stars know as magnetars, see, for example [6]. For magnetar cores made of quark matter the interior field can be estimated to

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reach the values $B\sim 10^{20}G$ [7]. The possibility to create a strong quasi-uniform magnetic field with the strength of the hadronic scale $B\sim 10^{19}G$, or even higher in heavy-ion collisions at RHIC and LHC, when the matter in the central region is presumably in the QGP phase, is recently shown [8]. Superconducting cosmic string - if they exist - could generate fields more then $10^{30}G$ in their vicinities [9].

Recently, the Schwinger effective action approach [10] was formally applied to calculate the probability for the vacuum to remain a vacuum in a linearly growing magnetic field for neutral fermions of spin 1/2 with anomalous magnetic moment. The same problem in $2+1$ dimensions was considered in [11], in $3+1$ dimensions in [12]. It is difficult to accept results presented in [12], which, in particular, admits neutral particle creation in a homogeneous magnetic field. This means that formal calculations a la Schwinger, without any theoretical justification based on quantum field theory (QFT), can lead to mistakes. Results of the work [11] seem to be reasonable, but essentially use specific of gamma matrices in $2+1$ dimensions, and cannot provide complete description of the effect.

It should be noted that until now a consistent description (based on QFT) of particle creation (due to their magnetic moments) in strong inhomogeneous magnetic fields was unknown. To provide such a description is a part of the present article. In the framework of the developed approach, we demonstrate that in a specific cases, the problem can be technically reduced to the problem of charge particle creation by an electric field given by a step scalar potential. We analyze once again the latter problem, using general QFT formulation of particle creation by potential steps presented recently, and formulating for the first time its solution from the view point of QFT. As a final result, we calculate neutral particle creation from the vacuum by a linearly growing magnetic field.

2 Dirac-Pauli equation with a constant magnetic field

In $3+1$ dimensions (dim.) and in the relativistic quantum mechanics neutral fermions of spin 1/2 and mass m with anomalous magnetic moment μ (without electric dipole moment) in an external electromagnetic field $F_{\lambda\nu}$ are described the Dirac-Pauli equation, see [14, 15]. Such an equation has the form[1]:

$$\left(\gamma^{\lambda}\hat{p}_{\lambda}-m-\frac{1}{2}\mu\sigma^{\lambda\nu}F_{\lambda\nu}\right)\psi\left(x\right)=0\,,$$
$$\hat{p}_{\nu}=i\partial_{\nu},\ \sigma^{\lambda\nu}=\frac{i}{2}\left[\gamma^{\lambda},\gamma^{\nu}\right],\tag{1}$$

where $F_{\lambda\nu}\left(x\right)$ is the field tensor, $\psi\left(x\right)$ is a four spinor, $x=\left(x^{0}=t,\mathbf{r}\right)$, $\mathbf{r}=\left(x,y,z\right)$, and $\gamma^{\nu}=\left(\gamma^{0},\boldsymbol{\gamma}\right)$ are Dirac matrices.

Let the external field is a constant nonuniform magnetic field $\mathbf{B}$ that is directed along the z-axis and depends on the coordinate y only, $\mathbf{B}\left(y\right)=\left(0,0,B_{z}\left(y\right)\right)$ such that the only nonzero components of the stress tensor are $F_{21}\left(y\right)=-F_{12}\left(y\right)=B_{z}\left(y\right)$. In addition, we suppose that $B_{z}\left(y\right)$ takes constant values as $y\rightarrow\pm\infty$, such that

$$\partial_{y}B_{z}\left(y\right)\overset{y\rightarrow\pm\infty}{\longrightarrow}0.$$

Moreover, we suppose that for $y<y_{L}$ (the range $S_{L}=\left(-\infty,y_{L}\right]$) and $y>y_{R}$ (the range $S_{R}=\left[y_{R},\infty\right)$) the field $B_{z}\left(y\right)$ is already uniform and its values are $B_{z}\left(y\right)=B_{z}\left(-\infty\right)$ and $B_{z}\left(y\right)=B_{z}\left(+\infty\right)$, respectively. Thus, the magnetic field under consideration is constant uniform (or zero) at spatial infinities and, in fact, represents either a potential barrier or

[1]Here we are using the natural system of units $\hbar=c=1$.

step for the magnetic moment μ. With such an external field, equation (1) takes the form:

$$i\partial_0\psi\left(t,\mathbf{r}\right)=\hat{H}\psi\left(t,\mathbf{r}\right)\ ,\quad \hat{H}=\left(\gamma^0\gamma^3\hat{p}^3+\gamma^0\Sigma_z\hat{\Pi}_z\right)\ ,$$
$$\hat{\Pi}_z=\Sigma_z\boldsymbol{\gamma}\widehat{\mathbf{p}}_\perp+m\Sigma_z-\mu B_z\left(y\right),\ \widehat{\mathbf{p}}_\perp=\left(\hat{p}^1,\hat{p}^2,0\right). \tag{2}$$

In the case under consideration, the operators $\hat{p}^0$, $\hat{p}^1$, $\hat{p}^3$, and $\hat{\Pi}_z$ are mutually commuting integrals of motion (all these operators commute with the Hamiltonian $\hat{H}$). The integral of motion $\hat{\Pi}_z$ is generalization of z-component of a spin polarization tensor for a uniform magnetic field, see [15].

It is useful to use an additional spin operator $\hat{R}$, which is also an integral of motion commuting with the previous ones,

$$\hat{R}=\hat{H}\hat{\Pi}_z^{-1}\left[1+\left(\hat{p}^3\hat{\Pi}_z^{-1}\right)^2\right]^{-1/2}. \tag{3}$$

A complete set of solutions of equation (2) can be written in the form

$$\psi_n\left(t,\mathbf{r}\right)=\exp\left(-ip_0t+ip_xx+ip_zz\right)\psi_n\left(y\right), \tag{4}$$

where $\psi_n\left(y\right)$ are eigenvectors of the equations

$$s\omega\sqrt{1+\left(p_z/\omega\right)^2}R\psi_n\left(y\right)=p_0\psi_n\left(y\right)\Longrightarrow R\psi_n\left(y\right)=s\psi_n\left(y\right),\ p_0=\omega\sqrt{1+\left(p_z/\omega\right)^2},$$
$$R=\left[1+\left(p_z/\omega\right)^2\right]^{-1/2}\left(s\gamma^0\gamma^3p_z/\omega+\gamma^0\Sigma_z\right), \tag{5}$$
$$\left[\hat{\Pi}_z\left(p_x,y\right)-s\omega\right]\psi_n\left(y\right)=0,\ s=\pm1,$$
$$\hat{\Pi}_z\left(p_x,y\right)=\hat{\pi}_z-\mu B_z\left(y\right),\ \hat{\pi}_z=\Sigma_z\left(\gamma^1p_x+\gamma^2\hat{p}^2\right)+m\Sigma_z, \tag{6}$$

and $n=\left(p_x,p_z,\omega,s\right)$ is the set of quantum numbers from a complete set of numbers that will be specified in which follows. Choosing $\psi_n\left(y\right)$ as

$$\psi_n\left(y\right)=\frac{1}{2}\left(1+sR\right)\Phi\left(y\right),$$

where $\Phi\left(y\right)$ is an arbitrary spinor, we obey eq. (5). The real continuous quantum number ω can be positive and negative and determines the transversal part of full energy, $\omega^2=p_0^2-p_z^2$, that is, it determines the full energy of particle moving on xy plane.

Then solutions of eq. (6) can be represented as

$$\psi_n\left(y\right)=\frac{1}{2}\left(1+sR\right)\left[\hat{\pi}_z+\mu B_z\left(y\right)+s\omega\right]\phi_n\left(y\right), \tag{7}$$

where the spinors $\phi_n\left(y\right)$ satisfy the following equation:

$$\left\{-\partial_y^2+m^2+p_x^2-\mu\gamma^1\partial_yB_z\left(y\right)-\left[\omega+s\mu B_z\left(y\right)\right]^2\right\}\phi_n\left(y\right)=0. \tag{8}$$

It is convenient to represent the spinor $\phi_n\left(y\right)$ in the form

$$\phi_n\left(y\right)=\varphi_{n,\chi}\left(y\right)\frac{1}{2}\left(1+i\chi\gamma^1\right)v, \tag{9}$$

where it is selected that either $\chi=+1$ or $\chi=-1$, v is an arbitrary constant spinor, and scalar functions $\varphi_{n,\chi}\left(y\right)$ are solutions of the equation

$$\left\{-\partial_y^2+m^2+p_x^2+i\chi\mu\partial_yB_z\left(y\right)-\left[\omega+s\mu B_z\left(y\right)\right]^2\right\}\varphi_{n,\chi}\left(y\right)=0. \tag{10}$$

In what follows, we suppose that v is normalized as $v^\dagger v = 1$. In addition, $vv^\dagger$ is the identity 4×4 matrix, $vv^\dagger = I$. Thus, the spinor structure of solutions (7) is defined completely. One can easily verify that solutions (7) that differ by values of χ only, are linearly dependent, this is an effect which the projection operator $[\ldots]$ in the representation (7) produces. Because of this, it is enough to work with solutions corresponding to one of possible two χ. That is why, the upperscript χ will sometimes disappear from solutions, but in such cases it is supposed that χ is fixed in a certain way, the same for all solutions under consideration.

Using the freedom inherent in the solutions of eq. (10), we construct two (in general different) sets $\{{}_\zeta\psi_n(t,\mathbf{r})\}$ and $\{{}^\zeta\psi_n(t,\mathbf{r})\}$ of independent solutions, $\zeta = \pm$, satisfying specific boundary conditions as $y \to -\infty$ or $y \to +\infty$. The first set contains states ${}_\zeta\psi_n(t,\mathbf{r})$ with definite real values p^L of y-component of the momentum, such that ζ defines the sign of the momentum,

$$-i\partial_y\ {}_\zeta\psi_n(t,\mathbf{r}) = \ p^L\ {}_\zeta\psi_n(t,\mathbf{r}),\ \zeta = \operatorname{sgn}\, p^L,\ y \to -\infty. \tag{11}$$

The second set contains states ${}^\zeta\psi_n(t,\mathbf{r})$ with definite real values p^R of the y-component of the momentum, and again ζ defines the sign of the momentum,

$$-i\partial_y\ {}^\zeta\psi_n(t,\mathbf{r}) = \ p^R\ {}^\zeta\psi_n(t,\mathbf{r}),\ \zeta = \operatorname{sgn}\, p^R,\ y \to +\infty. \tag{12}$$

We are interested in the nondecaying solutions of eq. (10) as $y \to \pm\infty$. In this case both p^L and p^R are real. We believe that for any given quantum numbers n both set $\{{}_\zeta\psi_n(t,\mathbf{r})\}$ and $\{{}^\zeta\psi_n(t,\mathbf{r})\}$ represent complete set of nondecaying solutions. In fact this is the above mentioned supposition about the form of the field $B_z(y)$.

It should be noted that the time independence of the magnetic field under consideration is an idealization. In fact, it is supposed that a field inhomogeneity was switched on in a time instant t_{in} then acts as the constant field during a large time T, and was switched off in a time instant $t_{out} = t_{in} + T$, and one can ignore effects of its switching on and off. This is a kind of the regularization, which could be, under certain conditions, replaced by periodic in t boundary conditions. Namely, by analogy with periodic boundary conditions in space, that are usually imposed as the volume regularization, here we impose periodic (with the period T) boundary condition in time t. Thus, we consider a theory in a big 3-dimensional space-time box that has a volume $V_y = TS_{xz}$, $S_{xz} = L_x \times L_z$, where L_x, L_z, and T are macroscopically large, $L_x, L_z \to \infty$ and $T \to \infty$.

It is convenient to use the inner product on the time-like hyperplane $y = \text{const}$, which has the form

$$(\psi, \psi')_y = \int_{V_y} \psi^\dagger(t,\mathbf{r})\,\gamma^0\gamma^2\psi'(t,\mathbf{r})\,dtdxdz. \tag{13}$$

The integration in (13) is fulfilled in the limits from $-L_x/2$ to $+L_x/2$, $-L_y/2$ to $+L_y/2$, and from $-T/2$ to $+T/2$ in the time t. It is supposed that all the functions ψ are periodic under translations from one box to another one. Under these suppositions, inner product (13) does not depend on y. We note that the quantity (13) for $\psi' = \psi$ represents the particle current via the hyperplane $y = \text{const}$.

By using the inner product (13), we obtain:

$$(\psi_n, \psi'_k)_y = V_y\delta_{n,k}\psi_n^\dagger(y)\,\gamma^0\gamma^2\psi'_n(y).$$

Thus, the current density in y-direction in the state $\psi_n(t,\mathbf{r})$ is

$$\mathcal{I}_n = \psi_n^\dagger(y)\,\gamma^0\gamma^2\psi_n(y). \tag{14}$$

Using the structure (7), we rewrite the combination $\psi_n^\dagger(y)\,\gamma^0\gamma^2\psi'_n(y)$ as follows

$$\varphi^*_{n,\chi}(y)\,\mathrm{tr}\left\{\left[-\Sigma_z\left(\gamma^1 p_x + i\overleftarrow{\partial}_y\gamma^2\right) - m\Sigma_z - \mu B_z(y) - s\omega\right]\gamma^0\gamma^2\frac{1}{2}[1+sR]\right.$$
$$\left.\left[-\Sigma_z\left(\gamma^1 p_x - i\overrightarrow{\partial}_y\gamma^2\right) - m\Sigma_z - \mu B_z(y) - s\omega\right]\frac{1}{2}\left(1+\chi i\gamma^1\right)\right\}\varphi'_{n,\chi}(y),$$

where tr$\{...\}$ is the trace in the space of 4×4 matrices. Calculating this trace, we obtain

$$\psi_n^\dagger(y)\gamma^0\gamma^2\psi_n'(y) = \left(1+(p_z/\omega)^2\right)^{-1/2}\varphi_{n,\chi}^*(y)\left(i\overleftarrow{\partial}_y - i\overrightarrow{\partial}_y\right)\left(\omega + s\mu B_z(y) + s\chi i\overrightarrow{\partial}_y\right)\varphi_{n,\chi}'(y)\,. \tag{15}$$

As was already mentioned, we supposed that $B_z(y)$ tends to some constants values as $y\to\pm\infty$. Let us suppose for definiteness sake that the derivative $\partial_y B_z(y)$ has a definite sign, let say $\partial_y B_z(y)\geq 0$, $\forall y$, and let $\mu<0$. Note that there are no bound states in this case. To simplify the consideration, we also suppose that

$$U = U_R - U_L > 0,\ U_L = -\mu B_z(-\infty) < 0,\ U_R = -\mu B_z(+\infty) > 0.$$

For asymptotic (as $|y|\to\infty$) states with real values p^L and/or p^R, we have

$${}_\zeta\varphi_{n,\chi}(y) = {}_\zeta\mathcal{N}\exp\left(ip^L y\right),\ \ {}^\zeta\varphi_n^{(\chi)}(x) = {}^\zeta\mathcal{N}\exp\left(ip^R y\right), \tag{16}$$

respectively, where ${}_\zeta\mathcal{N}$ and ${}^\zeta\mathcal{N}$ are normalization factors. We introduce the notation

$$\begin{aligned} E_s(L/R) &= \pi_s(L/R)\sqrt{1+[p_z/\pi_s(L/R)]^2}, \\ \pi_s(L/R) &= \omega - sU_{L/R}\ ,\ \pi_x = \sqrt{p_x^2+m^2} \end{aligned}$$

and note the following relations

$$\pi_s(L) = \pi_s(R) + sU \tag{17}$$

and

$$\left(p^L\right)^2 = [E_s(L)]^2 - \pi_x^2 - p_z^2,\ \left(p^R\right)^2 = [E_s(R)]^2 - \pi_x^2 - p_z^2, \tag{18}$$

where the last hold due to eq. (10). We see that $|E_s(L)|$ and $|E_s(R)|$ are the asymptotic values of kinetic energy, while $|\pi_s(L)|$ and $|\pi_s(R)|$ are the asymptotic values of its transversal part, respectively.

Then, using the asymptotic conditions (11), (12), and the result (15), we can subject the introduced sets $\{{}_\zeta\psi_n(t,\mathbf{r})\}$ and $\left\{{}^\zeta\psi_n(t,\mathbf{r})\right\}$ to the following orthonormality conditions

$$\left({}_\zeta\psi_n, {}_{\zeta'}\psi_{n'}\right)_x = \zeta\eta_L\delta_{\zeta,\zeta'}\delta_{n,n'};\ \ \left({}^\zeta\psi_n, {}^{\zeta'}\psi_{n'}\right)_x = \zeta\eta_R\delta_{\zeta,\zeta'}\delta_{n,n'}, \tag{19}$$

where

$$\eta_L = \mathrm{sgn}\pi_s(L)\,,\quad \eta_R = \mathrm{sgn}\pi_s(R)\,.$$

In deriving (19), it was taken into account that for asymptotic (as $|y|\to\infty$) states with real values p^L and p^R, the relations

$$|\pi_s(L)| > \left|p^L\right|,\ |\pi_s(R)| > \left|p^R\right|$$

hold due (18), respectively. That is why the sign of the quantity (15) with the operator $\left[\pi_s(L/R) + s\chi i\overrightarrow{\partial}_y\right]$ is due to the sign of the $\pi_s(L/R)$. The normalization factors in (16) are as follows

$$\begin{aligned} {}_\zeta\mathcal{N} &= {}_\zeta CY,\ {}^\zeta\mathcal{N} = {}^\zeta CY,\ Y = \left(1+(p_z/\omega)^2\right)^{1/4}V_y^{-1/2}, \\ {}_\zeta C &= \left[2\left|p^L\right|\left|\pi_s(L) - s\chi p^L\right|\right]^{-1/2},\ \ {}^\zeta C = \left[2\left|p^R\right|\left|\pi_s(R) - s\chi p^R\right|\right]^{-1/2}. \end{aligned} \tag{20}$$

In the limit of infinite volume of the normalization (continuous momenta p_0, p_x, and p_z) one has to substitute $\delta_{n,n'}$ in normalization conditions (19) by $\delta_{s,s'}\delta(p_0-p_0')\,\delta(p_x-p_x')\,\delta(p_z-p_z')\,.$ In this case, $V_y^{-1/2}\to(2\pi)^{-3/2}$ in eqs. (20).

For any given quantum numbers n, both set $\{{}_\zeta\psi_n(t,\mathbf{r})\}$ and $\{{}^\zeta\psi_n(t,\mathbf{r})\}$ represent complete set of nondecaying solutions of equation (2). Their mutual decompositions have the form:

$$\begin{aligned} \eta_L\ {}^\zeta\psi_n(t,\mathbf{r}) &= {}_+\psi_n(t,\mathbf{r})\, g\left({}_+\middle|{}^\zeta\right) - {}_-\psi_n(t,\mathbf{r})\, g\left({}_-\middle|{}^\zeta\right); \\ \eta_R\ {}_\zeta\psi_n(t,\mathbf{r}) &= {}^+\psi_n(t,\mathbf{r})\, g\left({}^+\middle|{}_\zeta\right) - {}^-\psi_n(t,\mathbf{r})\, g\left({}^-\middle|{}_\zeta\right), \end{aligned} \tag{21}$$

where the decomposition coefficients g are defined by the relations:

$$\left({}_\zeta\psi_n, {}^{\zeta'}\psi_{n'}\right)_y = \delta_{nn'}\, g\left({}_\zeta\middle|{}^{\zeta'}\right),\ g\left({}^{\zeta'}\middle|{}_\zeta\right) = g\left({}_\zeta\middle|{}^{\zeta'}\right)^*. \tag{22}$$

Using the orthonormality conditions (19), we derive the following relations for the decomposition coefficients:

$$\begin{aligned} g\left({}^{\zeta'}\middle|{}_+\right) g\left({}_+\middle|{}^\zeta\right) - g\left({}^{\zeta'}\middle|{}_-\right) g\left({}_-\middle|{}^\zeta\right) &= \zeta\eta_L\eta_R\delta_{\zeta,\zeta'}; \\ g\left({}_{\zeta'}\middle|{}^+\right) g\left({}^+\middle|{}_\zeta\right) - g\left({}_{\zeta'}\middle|{}^-\right) g\left({}^-\middle|{}_\zeta\right) &= \zeta\eta_L\eta_R\delta_{\zeta,\zeta'}. \end{aligned} \tag{23}$$

In particular, these relations imply that

$$\left|g\left({}_-\middle|{}^+\right)\right|^2 = \left|g\left({}_+\middle|{}^-\right)\right|^2,\ \left|g\left({}_+\middle|{}^+\right)\right|^2 = \left|g\left({}_-\middle|{}^-\right)\right|^2,\ \frac{g\left({}_+\middle|{}^-\right)}{g\left({}_-\middle|{}^-\right)} = -\frac{g\left({}^+\middle|{}_-\right)}{g\left({}^+\middle|{}_+\right)}. \tag{24}$$

Thus, one can see that all these coefficients can be expressed via only two of them, e.g. via $g\left({}_+\middle|{}^+\right)$ and $g\left({}_+\middle|{}^-\right)$. However, even these coefficients are not completely independent, they are related as follows:

$$\left|g\left({}_+\middle|{}^-\right)\right|^2 - \left|g\left({}_+\middle|{}^+\right)\right|^2 = -\eta_L\eta_R. \tag{25}$$

3 Creation of neutral fermions

It is useful to make a preliminary qualitative analysis of the behavior of particles and antiparticles in the fields under consideration. It should be noted that here there exist two principally different cases, the first one corresponds to $U < 2m$, whereas the second one (we call it creation case, C-case) corresponds to $U > 2m$. In the first case, there exist only a scattering of neutral fermions by the magnetic field without additional particle creation from the vacuum. This case can be treated in the framework of one-particle relativistic quantum mechanics. The quantum number s gives the spin polarization both for particles and antiparticles. Choosing the magnetic moment of the particle as μ, we have the magnetic moment of the antiparticle as $-\mu$. Note that we fix $\mu = -|\mu|$. Then, according to the standard particle-antiparticle identification of wave functions, asymptotic kinetic energy (at $y \to \pm\infty$) of the particle moving on xy plane is $\pi_s(L/R) > 0$, while it is $-\pi_s(L/R) > 0$ for the antiparticle. One can see from eq. (17) that the particle potential energy $s|\mu|B_z(y)$ decreases along the axis y for $s = -1$ and increases for $s = +1$. At the same time, the antiparticle potential energy $-s|\mu|B_z(y)$ increase along y for $s = -1$ and decrease for $s = +1$. That means that the field $B_z(y)$ accelerates particles with $s = -1$ and antiparticle with $s = +1$ along the axis y. Respectively, antiparticles with $s = -1$ and particles with $s = +1$ are accelerated by the field in the opposite direction. The same observation holds in the case $U > 2m$.

We note that real particles are described by some localized in the space-time wave packets, such that we have to study the motion of such packets in the external field (obviously, it is enough to speak about a localization in the y-direction). Let us denote by $S_{\rm int}$ the region, where the magnetic field is inhomogeneous. In the region S_L, situated left from $S_{\rm int}$ and in the region S_R right from $S_{\rm int}$, the magnetic field is homogeneous. For big enough differences

U between the initial and final potential energies, particles and antiparticles with any initial kinetic momenta along the axis y get final kinetic momenta directed always as their the acceleration by the magnetic field. That is what we have in the case $U > 2\sqrt{p_x^2 + p_z^2 + m^2}$ for all partial waves with given p_x and p_z of a wave packet. Because particles and their antiparticles with a given s have opposite directions of the acceleration, there exists a state polarization out of the region $S_{\rm int}$. The final particles with $s = +1$ and antiparticles with $s = -1$ are situated in the region S_L, and final antiparticles with $s = +1$ and particles with $s = -1$ are situated in the region S_R.

From the physical point of view, there is a similarity between the two cases, one where neutral fermions with anomalous magnetic moment are placed in an inhomogeneous magnetic field $B_z(y)$ with $\partial_y B_z(y) > 0$, and another one where charged fermions are placed in a constant electric field directed along y and given by a scalar potential $A_0(y)$. In both cases external fields produce a work which implies an acceleration of the corresponding particles in y-direction. From the QFT point of view if such a work is greater than $2m$ (C-case), a particle creation from the vacuum is possible. In fact, this analogy allows in both cases formally to use the same techniques of calculation. It turns out that the problem of neutral fermion creation in strong inhomogeneous magnetic field can be technically reduced to the problem of charge particle creation by an electric potential step. Some heuristic exact calculations of the particle creation by potential steps in the framework of the relativistic quantum mechanics were presented by Nikishov [16, 17], developed then in [18], and were used in numerous works in the frame work of semiclassical considerations, see reviews [19, 20].

In such a way it seems that we could use the known results to find the mean number of neutral particle-antiparticle created. However, a closer consideration shows that the particle-antiparticle and causal identification of wave functions ${}^{\zeta}\psi_n(t, \mathbf{r})$ and ${}_{\zeta}\psi_n(t, \mathbf{r})$ given by Nikishov [16, 17] does not coincide with one given by Hansen and Ravndal [18] for the C-case, see discussion in [21]. Within the WKB approximation this difficulty can be bypassed, but the question remains. Trying to resolve this contradiction, we have realized that at that time no any justification for quantum mechanical calculations from the QFT point of view were elaborated. Such a justification is given in our recent work [22] as a part of general QFT formulation of particle creation by potential steps. We follow the principle ideas of this work in solving the problem under consideration, taking into account some necessary identification.

In the C-case, there exists a range $2\sqrt{\pi_x^2 + p_z^2} < U$ of momentum p_x and p_z of the fermions, such that particle creation is possible. This case is described by wave functions (7) with quantum numbers of domain Ω, where ω, p_x, and p_z are restricted by the inequalities

$$s\pi_s(L) \geq \pi_x,\ s\pi_s(R) \leq -\pi_x,\ 2\sqrt{\pi_x^2 + p_z^2} < U. \tag{26}$$

If to treat this case using identification of wave function by an analogy with one-particle scattering theory, there appears an analog of Klein paradox for charged relativistic particles in an electric field [23]. This is an indication that one has to use appropriate many-particle description given by QFT to treat the problem correctly.

On the first stage of the canonical quantization of the field $\psi(t, \mathbf{r})$ one establishes that the corresponding quantum field is the Heisenberg field operator $\Psi(t, \mathbf{r})$ that satisfies the equal-time anticommutation relations:

$$[\Psi(t, \mathbf{r}), \Psi(t, \mathbf{r}')]_+ = \left[\Psi(t, \mathbf{r})^\dagger, \Psi(t, \mathbf{r}')^\dagger\right]_+ = 0,\ \left[\Psi(t, \mathbf{r}), \Psi(t, \mathbf{r}')^\dagger\right]_+ = \delta(\mathbf{r} - \mathbf{r}') \tag{27}$$

and the the Dirac-Pauli equation (2). The formal expressions for the Hamiltonian $\widehat{\mathcal{H}}$ of the quantized fermion field and the corresponding magnetic momentum operator $\widehat{\mathcal{M}}$ can be easily constructed,

$$\widehat{\mathcal{H}} = \int \Psi(t, \mathbf{r})^\dagger \hat{H} \Psi(t, \mathbf{r})\, d\mathbf{r},\ \widehat{\mathcal{M}} = \frac{\mu}{2} \int \left[\Psi(t, \mathbf{r})^\dagger, \Psi(t, \mathbf{r})\right]_- d\mathbf{r}. \tag{28}$$

To perform quantization in terms of particles and antiparticles, we define the following inner product

$$\left(\psi,\psi'\right)_t=\int_t \psi^\dagger\left(t,\mathbf{r}\right)\psi'\left(t,\mathbf{r}\right)d\mathbf{r} \tag{29}$$

between two solutions of the the Dirac-Pauli equation on t-const hyperplane. This inner product does not depends on the choice of such a hyperplane if the spinors $\psi\left(t,\mathbf{r}\right)$ obey certain boundary conditions that allow one to integrate by parts in (29) neglecting boundary terms. Since physical states are wave packets that vanish on the remote boundaries, the above supposition holds true and the inner product (29) is time independent for such states. Considering plane waves instead of natural wave packets, one has to impose corresponding periodic boundary conditions on the corresponding wave functions and external field to maintain the inner product (29) time independent. However, in the case under consideration the external field with different asymptotics at $y\to\pm\infty$ cannot be adopted to any periodic boundary conditions in y-direction without changing its physical meaning. Here to provide time independence of the inner product, one has to modify the inner product itself. This modification is applied to the integration over y in the expression (29) and is described below.

Let $\psi_n\left(t,\mathbf{r}\right)$ and $\psi'_{n'}\left(t,\mathbf{r}\right)$ be wave functions (7) and the integral over the variable y in the infinite limits be regularized by large positive numbers L_1 and L_2. Integrating over the variables x, y, and using representation (9), we obtain

$$\begin{aligned}&\left(\psi_n,\psi'_{n'}\right)_t=\delta_{n,n'}S_{xz}\mathcal{R},\ \ \mathcal{R}=\int_{-L_1}^{L_2}Qdy,\\ &Q=\left(\varphi_{n,\chi}\left(y\right)\right)^*\left[\pi_x^2+\left(\omega+s\mu B_z\left(y\right)+s\chi i\partial_y\right)^2\right]\varphi'_{n,\chi}\left(y\right),\end{aligned} \tag{30}$$

where the orthogonality for $n\neq n'$ follows as $L_1,\, L_2\to\infty$.

We represent the regularized integral $\mathcal{R}$ as follows

$$\mathcal{R}=\int_{-L_1}^{y_L}Qdy+\int_{y_L}^{y_R}Qdy+\int_{y_R}^{L_2}Qdy, \tag{31}$$

where only the second term, integral over the region $S_{\rm int}$, depends on the derivative $\partial_y B_z(y)$. The smoothness of the $\partial_y B_z(y)$ allows us to believe that this integral is finite as $L_1,\, L_2\to\infty$. The first and the third terms are calculated as integrals over the regions where $\partial_y B_z(y)=0$. Then their values are determined by asymptotics (16) in the following form

$$\begin{aligned}\mathcal{R}_L&=\int_{-L_1}^{y_L}Q_L dy,\ \ \mathcal{R}_R=\int_{y_R}^{L_2}Q_R dy,\\ Q_{L/R}&=\left(\varphi_{n,\chi}\left(y\right)\right)^*\left[\pi_x^2+\left(\pi_s\left(L/R\right)+s\chi i\partial_y\right)^2\right]\varphi'_{n,\chi}\left(y\right).\end{aligned} \tag{32}$$

Q_L and Q_R are constant then $\mathcal{R}_L\sim L_1$ and $\mathcal{R}_R\sim L_2$. We see that only $\mathcal{R}_L$ and $\mathcal{R}_R$ are essential in (31) as $L_1,\, L_2\to\infty$,

$$\mathcal{R}\underset{L_1,L_2\to\infty}{\longrightarrow}\mathcal{R}_L+\mathcal{R}_R\ .$$

There exist two independent solutions with given quantum number n of domain Ω. In spite of the fact that these solutions are obtained in the constant external field we believe that they represent asymptotic forms of some unknown solutions of the Dirac-Pauli equation with an external field $\partial_y B_z(t,y)$ that is switched on and off at $t\to\pm\infty$ and effects of the switching on and off are negligible. Since the inner product (29) does not depend on t for such solutions, we believe that orthogonal pairs of solutions that describe alternative

particle/antiparticle states at the initial and the final time instants remain orthogonal at arbitrary time instant. Therefore we have to find out which solutions among those we have introduced before are such orthogonal pairs. Taking into account the relations (21), one can show that

$$\left({}_{\zeta}\psi_n, {}^{\zeta}\psi_n\right)_t = 0, \quad n \in \Omega, \tag{33}$$

if we assume that L_1 and L_2 satisfy the relation

$$L_1 \left|\frac{\pi_s(L)}{p^L}\right| - L_2 \left|\frac{\pi_s(R)}{p^R}\right| = O(1). \tag{34}$$

Condition (34) guarantees that the sets $\{{}_{\zeta}\psi_n(t,\mathbf{r})\}$ and $\{{}^{\zeta}\psi_n(t,\mathbf{r})\}$ for $n \in \Omega$ correspond to alternative physical states.

Consider the quantities $\mathcal{R}_{L/R}$ (32) defined by the functions ${}_{\zeta}\varphi_n(x)$ and ${}^{\zeta}\varphi_n(x)$ with quantum numbers of domain Ω. In this case we attribute the corresponding index ζ to these quantities as follows: $\mathcal{R}_{L/R} \to {}_{\zeta}\mathcal{R}_{L/R}$ or $\mathcal{R}_{L/R} \to {}^{\zeta}\mathcal{R}_{L/R}$. Using eqs. (16) and (20) and retaining only leading in the limit $L_1, L_2 \to \infty$ terms, we obtain

$${}_{\zeta}\mathcal{R}_L = Y^2 L_1 \left|\frac{\pi_s(L)}{p^L}\right|, \quad {}^{\zeta}\mathcal{R}_R = Y^2 L_2 \left|\frac{\pi_s(R)}{p^R}\right|. \tag{35}$$

To calculate the quantities ${}_{\zeta}\mathcal{R}_R$ and ${}^{\zeta}\mathcal{R}_L$, we use relations (21). Again retaining only leading in the limit $L_1, L_2 \to \infty$ terms (neglecting in particular oscillating terms) and taking into account eqs. (16) and (20), we find

$$\begin{aligned} {}_{\zeta}\mathcal{R}_R &= Y^2 L_2 \left|\frac{\pi_s(R)}{p^R}\right| \left[\left|g\left({}_{\zeta}|^{+}\right)\right|^2 + \left|g\left({}_{\zeta}|^{-}\right)\right|^2\right], \\ {}^{\zeta}\mathcal{R}_L &= Y^2 L_1 \left|\frac{\pi_s(L)}{p^L}\right| \left[\left|g\left({}_{+}|^{\zeta}\right)\right|^2 + \left|g\left({}_{-}|^{\zeta}\right)\right|^2\right]. \end{aligned} \tag{36}$$

Note that ${}^{\zeta}\mathcal{R}_L > {}^{\zeta}\mathcal{R}_R$ and ${}_{\zeta}\mathcal{R}_R > {}_{\zeta}\mathcal{R}_L$ due to $\left|g\left({}_{+}|^{-}\right)\right|^2 > 1$. Taking unitarity relations (25) and condition (34) into account, we obtain the following orthonormality relations

$$\begin{aligned} &\left({}_{\zeta}\psi_n, {}_{\zeta}\psi_{n'}\right)_t = \delta_{n,n'}\mathcal{M}, \quad \left({}^{\zeta}\psi_n, {}^{\zeta}\psi_{n'}\right)_t = \delta_{n,n'}\mathcal{M}, \\ &\mathcal{M} = 2\frac{L_2}{T}\left|\frac{\pi_s(R)}{p^R}\right| \left|g\left({}_{+}|^{-}\right)\right|^2. \end{aligned} \tag{37}$$

One can see that the following symmetry take place: Particles with opposite values of s have opposite accelerations, the same is valid for antiparticles. That is why the cases $s = +1$ and $s = -1$ differs only by opposite directions of all the motions, and respectively by the opposite dispositions of all the asymptotic ranges. Probabilities of all the processes are equal in both the cases. That is why it is enough to consider only one case, let say $s = +1$.

It is supposed that we know the complete set of the solutions of the Dirac-Pauli equation, parameterized by a set of quantum numbers n, on the hyperplane $t = \text{const}$. Then we can decompose the quantum Heisenberg field operator $\Psi(t,\mathbf{r})$ and its Hermitian conjugate $\Psi^{\dagger}(t,\mathbf{r})$ in this complete set using the inner product (29). Assuming that both set $\{{}_{+}\psi_n(t,\mathbf{r}), {}^{+}\psi_n(t,\mathbf{r})\}$ and $\{{}_{-}\psi_n(t,\mathbf{r}), {}^{-}\psi_n(t,\mathbf{r})\}$ represent the complete set of non-decaying solutions for the domain Ω, we introduce the notation $\Psi_n(t,\mathbf{r})$ for the component of the quantum field operator that can be expanded via either ${}_{+}\psi_n(t,\mathbf{r}), {}^{+}\psi_n(t,\mathbf{r})$ or ${}_{-}\psi_n(t,\mathbf{r}), {}^{-}\psi_n(t,\mathbf{r})$. Operator coefficients in such decompositions do not depend on space-time coordinates because both quantum field operators and classical solutions obey the same Pauli-Dirac equation. For example, for $s = +1$, we can decompose the $\Psi_n(t,\mathbf{r})$ and $\Psi_n^{\dagger}(t,\mathbf{r})$ as follows

$$\begin{aligned} &\Psi_n(t,\mathbf{r}) = \mathcal{M}^{-1/2}\left[a_n(\text{out})\ {}^{+}\psi_n(t,\mathbf{r}) + b_n^{\dagger}(\text{out})\ {}_{+}\psi_n(t,\mathbf{r})\right], \\ &\Psi_n^{\dagger}(t,\mathbf{r}) = \mathcal{M}^{-1/2}\left[a_n^{\dagger}(\text{out})\ {}^{+}\psi_n^{\dagger}(t,\mathbf{r}) + b_n(\text{out})\ {}_{+}\psi_n^{\dagger}(t,\mathbf{r})\right]; \end{aligned} \tag{38}$$

and

$$\Psi_n(t,\mathbf{r}) = \mathcal{M}^{-1/2}\left[a_n(\text{in})\ {}^{-}\psi_n(t,\mathbf{r}) + b_n^\dagger(\text{in})\ {}_{-}\psi_n(t,\mathbf{r})\right],$$
$$\Psi_n^\dagger(t,\mathbf{r}) = \mathcal{M}^{-1/2}\left[a_n^\dagger(\text{in})\ {}^{-}\psi_n^\dagger(t,\mathbf{r}) + b_n(\text{in})\ {}_{-}\psi_n^\dagger(t,\mathbf{r})\right]. \tag{39}$$

In what follows, we interpret all a and b as annihilation and all $a^\dagger$ and $b^\dagger$ as creation operators; all a and $a^\dagger$ as describing particles and b and $b^\dagger$ as describing antiparticles; all the operators labeled by the argument in are in-operators, whereas all the operators labeled by the argument out are out-operators. It can be shown that these creation and annihilation operators obeying canonical anticommutation relations,

$$\begin{aligned}
[a_n(\text{in}), a_k^\dagger(\text{in})]_+ &= [a_n(\text{out}), a_k^\dagger(\text{out})]_+ = [b_n(\text{in}), b_k^\dagger(\text{in})]_+ = [b_n(\text{out}), b_k^\dagger(\text{out})]_+ = \delta_{nk},\\
[a_n(\text{in}), b_n(\text{in})]_+ &= [a_n(\text{in}), b_n^\dagger(\text{in})]_+ = [a_n(\text{out}), b_n(\text{out})]_+ = [a_n(\text{out}), b_n^\dagger(\text{out})]_+ = 0
\end{aligned} \tag{40}$$

due to relation (27). In such an interpretation, the in-vacuum $|0,\text{in}\rangle$ and out-vacuum $|0,\text{out}\rangle$ are defined by conditions,

$$a_n(\text{in})|0,\text{in}\rangle = b_n(\text{in})|0,\text{in}\rangle = 0,\ \forall n;$$
$$a_n(\text{out})|0,\text{out}\rangle = b_n(\text{out})|0,\text{out}\rangle = 0,\ \forall n. \tag{41}$$

Let consider magnetic momentum operator,

$$\widehat{\mathcal{M}}_\Omega = \frac{\mu}{2}\int\left[\Psi_\Omega(t,\mathbf{r})^\dagger, \Psi_\Omega(t,\mathbf{r})\right]_- d\mathbf{r} \tag{42}$$

and operator of kinetic energy of the quantum fermion field

$$\widehat{\mathcal{H}}_\Omega^{kin} = \int \Psi_\Omega(t,\mathbf{r})^\dagger\left[\hat{\Pi}_z + \mu B_z(y)\right]\sqrt{1+\left[\frac{\hat{p}^3}{\hat{\Pi}_z + \mu B_z(y)}\right]^2}\Psi_\Omega(t,\mathbf{r})\,d\mathbf{r} - H_\Omega^0 \tag{43}$$

in domain Ω, where $\Psi_\Omega(t,\mathbf{r}) = \sum_{n\in\Omega}\Psi_n(t,\mathbf{r})$ and $H_\Omega^0 = \left\langle 0,\text{in}\middle|\widehat{\mathcal{H}}_\Omega^{kin}\middle|0,\text{in}\right\rangle$ is the constant term corresponding to the energy of vacuum fluctuations. Using relations (35)-(37), (21), and (25), one can represent these operators in equivalent diagonal forms as follows

$$\begin{aligned}
\widehat{\mathcal{M}}_\Omega &= \mu\sum_{n\in\Omega}\left[a_n^\dagger(\text{in})a_n(\text{in}) - b_n^\dagger(\text{in})b_n(\text{in})\right]\\
&= \mu\sum_{n\in\Omega}\left[a_n^\dagger(\text{out})a_n(\text{out}) - b_n^\dagger(\text{out})b_n(\text{out})\right];\\
\widehat{\mathcal{H}}_\Omega^{kin} &= \sum_{n\in\Omega}\left[\ {}^{-}\mathcal{E}_n a_n^\dagger(\text{in})a_n(\text{in}) - \ {}_{-}\mathcal{E}_n b_n^\dagger(\text{in})b_n(\text{in})\right]\\
&\quad \sum_{n\in\Omega}\left[\ {}^{+}\mathcal{E}_n a_n^\dagger(\text{out})a_n(\text{out}) - \ {}_{+}\mathcal{E}_n b_n^\dagger(\text{out})b_n(\text{out})\right],
\end{aligned} \tag{44}$$

where

$${}_\zeta\mathcal{E}_n = E_{+1}(R) + \frac{1}{2}U\left|g\left({}_{+}|^{-}\right)\right|^{-2},\quad {}^\zeta\mathcal{E}_n = E_{+1}(L) - \frac{1}{2}U\left|g\left({}_{+}|^{-}\right)\right|^{-2},$$

see details in [22]. We suppose that in the external field under consideration we have

$${}^\zeta\mathcal{E}_n > 0,\quad {}_\zeta\mathcal{E}_n < 0, \tag{45}$$

so that the signs of energies ${}^\zeta\mathcal{E}_n$ and ${}_\zeta\mathcal{E}_n$ are determined by the signs of $\pi_{+1}(R/L)$. In known solvable cases inequalities (45) hold true, for example, see [16, 17, 21]. Thus, the

operator $\widehat{\mathcal{H}}_{\Omega}^{kin}$ is positive defined. This fact provides a consistent quantization in terms of particles and antiparticles in the domain Ω.

Kinetic energy must be positive for any wave packets of both particle and antiparticle. That is why particle wave packets are situated in the region S_L and antiparticle wave packets are situated in the region S_R, that is, there is the total reflection from $S_{\rm int}$ both for particles and antiparticles. This is consistent with the physical meaning. Note that the expressions $\left({}^{\zeta}\psi_n, {}^{\zeta'}\psi_{n'}\right)_x$ and $(-1)\left({}_{\zeta}\psi_n, {}_{\zeta'}\psi_{n'}\right)_x$, given by Eq. (19), are the probability currents of particles and antiparticles through the surface $y = const$, respectively. The particle and antiparticle currents are positive for $\zeta = -1$ and negative for $\zeta = +1$. Thus, we see that for $s = +1$ the functions ${}^{+}\psi_n(t, \mathbf{r})$ and ${}_{+}\psi_n(t, \mathbf{r})$ describe outgoing particles and antiparticles, while the functions ${}^{-}\psi_n(t, \mathbf{r})$ and ${}_{-}\psi_n(t, \mathbf{r})$ describe incoming particles and antiparticles, respectively. The particle-antiparticle and causal identification of wave functions (7) is unique in framework of QFT.

The vacuum corresponds to the absence of incoming particles and antiparticles. In such a case the presence of outgoing particles and antiparticles indicates particle creation from the vacuum. The effect of particle creation implies constant currents of outgoing particles and antiparticles. These currents are equal in the regions S_L and S_R.

Then taking into account eqs. (38) and (39), we obtain direct and inverse linear canonical transformations between the in and out creation and annihilation operators (Bogolyubov transformation):

$$
\begin{aligned}
& a_n(\mathrm{out}) = g\left({}^{-}|_{+}\right)^{-1} g\left({}^{+}|_{+}\right) a_n(\mathrm{in}) - g\left({}_{-}|^{+}\right)^{-1} b_n^{\dagger}(\mathrm{in}), \\
& b_n^{\dagger}(\mathrm{out}) = g\left({}^{-}|_{+}\right)^{-1} a_n(\mathrm{in}) + g\left({}_{-}|^{+}\right)^{-1} g\left({}_{+}|^{+}\right) b_n^{\dagger}(\mathrm{in}); \\
& a_n(\mathrm{in}) = g\left({}^{+}|_{-}\right)^{-1} g\left({}^{-}|_{-}\right) a_n(\mathrm{out}) + g\left({}_{+}|^{-}\right)^{-1} b_n^{\dagger}(\mathrm{out}). \\
& b_n^{\dagger}(\mathrm{in}) = -g\left({}^{+}|_{-}\right)^{-1} a_n(\mathrm{out}) + g\left({}_{+}|^{-}\right)^{-1} g\left({}_{-}|^{-}\right) b_n^{\dagger}(\mathrm{out}),
\end{aligned} \tag{46}
$$

These transformations are similar to that used by Nikishov in the problem of charge particle scattering on electric step [16, 17].

By the help of transformations (46), we calculate differential mean number of created particles and antiparticles

$$
\begin{aligned}
N_n^{(+)} &= \langle 0, \mathrm{in}| a_n^{\dagger}(\mathrm{out}) a_n(\mathrm{out}) |0, \mathrm{in}\rangle = \left|g\left({}_{-}|^{+}\right)\right|^{-2}, \\
N_n^{(-)} &= \langle 0, \mathrm{in}| b_n^{\dagger}(\mathrm{out}) b_n(\mathrm{out}) |0, \mathrm{in}\rangle = \left|g\left({}_{+}|^{-}\right)\right|^{-2}.
\end{aligned} \tag{47}
$$

Relations (24) imply the equality

$$
N_n^{(+)} = N_n^{(-)} = N_n,
$$

which allows us to treat N_n as differential mean number of created pairs. The total number N of created pairs is the sum

$$
N = \sum_{n \in \Omega} N_n. \tag{48}
$$

The elementary relative probability amplitudes of particle creation, annihilation and scattering are defined as follows

$$
\begin{aligned}
& w(+|+)_{n'n} = c_v^{-1} \langle 0, \mathrm{out}\left| a_{n'}(\mathrm{out}) a_n^{\dagger}(\mathrm{in}) \right| 0, \mathrm{in}\rangle, \\
& w(-|-)_{nn'} = c_v^{-1} \langle 0, \mathrm{out}\left| b_{n'}(\mathrm{out}) b_n^{\dagger}(\mathrm{in}) \right| 0, \mathrm{in}\rangle, \\
& w(0|-+)_{nn'} = c_v^{-1} \langle 0, \mathrm{out}\left| b_n^{\dagger}(\mathrm{in}) a_{n'}^{\dagger}(\mathrm{in}) \right| 0, \mathrm{in}\rangle, \\
& w(+-|0)_{n'n} = c_v^{-1} \langle 0, \mathrm{out}\left| a_{n'}(\mathrm{out}) b_n(\mathrm{out}) \right| 0, \mathrm{in}\rangle,
\end{aligned} \tag{49}
$$

where c_v is the vacuum-to-vacuum transition amplitude, $c_v = \langle 0, \mathrm{out}|0, \mathrm{in}\rangle$. Amplitudes (49) are diagonal

$$
\begin{aligned}
& w\left(+|+\right)_{n'n} = \delta_{n,n'} w_n\left(+|+\right), \; w\left(-|-\right)_{nn'} = \delta_{n,n'} w_n\left(-|-\right), \\
& w\left(0|-+\right)_{nn'} = \delta_{n,n'} w_n\left(0|-+\right), \; w\left(+-|0\right)_{n'n} = \delta_{n,n'} w_n\left(+-|0\right)
\end{aligned} \tag{50}
$$

and can be expressed via the coefficients $g\left({}^{\zeta'}|_{\zeta}\right)$ as follows:

$$
\begin{aligned}
& w_n\left(+|+\right) = g\left({}^{+}|_{-}\right) g\left({}^{-}|_{-}\right)^{-1} = g\left({}_{+}|^{-}\right) g\left({}_{+}|^{+}\right)^{-1}, \\
& w_n\left(-|-\right) = g\left({}^{-}|_{+}\right) g\left({}^{-}|_{-}\right)^{-1} = g\left({}_{-}|^{+}\right) g\left({}_{+}|^{+}\right)^{-1}, \\
& w_n\left(+-|0\right) = g\left({}_{+}|^{+}\right)^{-1}, \; w_n\left(0|-+\right) = -g\left({}^{-}|_{-}\right)^{-1},
\end{aligned} \tag{51}
$$

where transformations (46) are used.

One can express probabilities of particle scattering and a pair creation when $n \in \Omega$, and the probability for the vacuum to remain a vacuum via the differential mean numbers N_n as follows

$$
\begin{aligned}
& P(+|+)_{n,n'} = |<0, out|a_n(out)a^{\dagger}_{n'}(in)|0, in>|^2 = \delta_{n,n'}\frac{1}{1-N_n}P_v \;, \\
& P(-+|0)_{n,n'} = |<0, out|b_n(out)a_{n'}(out)|0, in>|^2 = \delta_{n,n'}\frac{N_n}{1-N_n}P_v \;, \\
& P_v = |c_v|^2 = \exp\left\{\sum_{n\in\Omega}\ln\left(1-N_n\right)\right\} \;,
\end{aligned} \tag{52}
$$

see details in [22]. The probabilities for an antiparticle scattering and a pair annihilation are described by the same expressions $P(+|+)$ and $P(-+|0)$, respectively.

4 Quasilinear magnetic field

Here, we consider a specific case of inhomogeneous magnetic field, namely a field linearly growing on an interval L_y. More exactly, the field of the form

$$
B_z(y) = \begin{cases} B_0, \; y < 0 \\ B_0 + B'y, \; y \in [0, L_y] \\ B_0 + B'L_y, \; y > L_y \end{cases} ,
$$

where $B' > 0$ and $B_0 = -\,B'L_y/2$. Let us call such field quasilinear magnetic field. Consider the case given by the condition

$$
\sqrt{|\mu B'|}L_y \gg \max\left\{1, m/\sqrt{|\mu B'|}\right\}, \tag{53}
$$

which implies that there is a particle creation in a wide enough range Ω of momenta given by condition (26). One can demonstrate, similar to the case considered in [25], see also [26], that leading contributions to the differential mean numbers N_n of created pairs do not depend on L_y in the limit $L_y \to \infty$. That is why, it is enough to consider the case of linearly growing magnetic field. Equation (10) in the latter field for the function $\varphi_{n,\chi}(y)$ given by (9) can be written as

$$
\begin{aligned}
& \left(\frac{d^2}{d\xi^2} + \xi^2 - \lambda + i\chi\right)\varphi_{n,\chi}(y) = 0, \\
& \xi = \sqrt{|\mu|\,B'}\left[y + \left(|\mu|\,B'\right)^{-1}\left(|\mu|\,B_0 - \omega\right)\right], \; \lambda = \frac{m^2 + p_x^2}{|\mu B'|}.
\end{aligned} \tag{54}
$$

Solutions of this equation, obeying boundary condition (11) and (12), have the forms

$$
{}_{+}^{-}\varphi_{n,\chi}\left(y\right)=\mathcal{N}_{\chi}D_{-\nu-1}\left[\pm(1+i)\xi\right], \;\; {}_{-}^{+}\varphi_{n,\chi}\left(y\right)=\mathcal{N}_{\chi}D_{\nu}\left[\pm(1-i)\xi\right], \tag{55}
$$

where $D_{\nu}\left(z\right)$ are Weber parabolic cylinder (WPC) functions, $\nu=-\left(i\lambda+1+\chi\right)/2$. By the help of an asymptotic expansion of WPC-functions, one can verify the validity of the boundary condition (11) and (12). Using solutions (55), we construct the sets $\left\{{}_{\zeta}\psi_{n}\left(t,\mathbf{r}\right)\right\}$ and $\left\{{}^{\zeta}\psi_{n}\left(t,\mathbf{r}\right)\right\}$ of solutions of the Dirac-Pauli equation.

The obtained form of solutions formally coincide with the one found in [16, 17, 26] for the case of charged particle creation by constant uniform electric field, compare with [25]. Note that our identification of wave functions is in agreement with one given by Nikishov for such a special case. This allows us to use these calculations to find differential mean numbers of created pairs given by eq. (47).

As a result, we obtain

$$
N_{n}=e^{-\pi\lambda} \tag{56}
$$

in the limit $\sqrt{|\mu B'|}L_{y}\gg K$ if the ω and p_{z} satisfies the following condition

$$
|\omega|<\omega_{\max},\;|p_{z}|<\omega_{\max},\;\omega_{\max}=\left|\mu B'\right|L_{y}/2-\sqrt{|\mu B'|}K,
$$

where K is a given arbitrary number $K\gg\max\left\{1,m/\sqrt{|\mu B'|}\right\}$. Following the idea of finite work regularization presented in [25], one can show that an exact expression for N_{n} is rapidly decreasing as $|\omega|\to\infty$ due to the finite work of this field, $\left|\mu B'\right|L_{y}$, that is, $\omega_{\max}$ is an effective maximum value of quantum number $|\omega|$ for the quasilinear field under consideration. The maximum value for $|p_{z}|$ of range Ω follows from condition (26). One can check that the mean numbers do not depend on the sign of $\mu B'$ and on the spin polarization s. Note, however, that unlike the case of particle creation due to the electric potential step, the neutral particle (antiparticle) created with different s form fluxes aimed in opposite directions. The leading approximation given by expression (56) does not depend on quantum numbers ω and p_{z}. Although result (56) has been derived for $B'=const$ field, it can be applicable to a spatially slowly varying $B'\left(y\right)$ as a good approximation if the gradient variation is sufficiently small.

Let us calculate the total number $\mathcal{N}_{s}$ of created pairs with given s defined by (48). To do this we go over to the integral

$$
\sum_{p_{x},p_{z},p_{0}}\ldots=\frac{L_{x}L_{z}T}{\left(2\pi\right)^{3}}\int dp_{x}dp_{z}dp_{0}\ldots\,.
$$

Taking into account that the exact distribution N_{n} plays the role of a cut-off factor in the integral over ω, p_{x}, and p_{z} we represent the total number $\mathcal{N}_{s}$ in the form

$$
\mathcal{N}_{s}=2\int_{0}^{\omega_{\max}}dp_{z}\mathcal{N}_{s,p_{z}},\;\;\mathcal{N}_{s,p_{z}}=\frac{L_{x}L_{z}T}{\left(2\pi\right)^{3}}\int dp_{x}\int_{0}^{\omega_{\max}^{2}}\frac{N_{n}d\omega^{2}}{\sqrt{\omega^{2}+p_{z}^{2}}}, \tag{57}
$$

where the relation $p_{0}=\omega\sqrt{1+\left(p_{z}/\omega\right)^{2}}$ from (5) is used. We obtain the leading contribution in (57) as follows

$$
\mathcal{N}_{s,p_{z}}=L_{x}L_{z}Tn_{s,p_{z}},\;\;n_{s,p_{z}}=\frac{\sqrt{|\mu B'|}}{4\pi^{3}}\exp\left(-\frac{\pi m^{2}}{|\mu B'|}\right)\left(\sqrt{\omega_{\max}^{2}+p_{z}^{2}}-|p_{z}|\right). \tag{58}
$$

From (58), we see that the leading term of density $n_{s,p_{z}}$ is linear function of the length L_{y} for sufficiently small momentum p_{z}, $|p_{z}|\ll\omega_{\max}$, that is, the density of the particles created per unit space-time volume, $n_{s,p_{z}}/L_{y}$, is uniform. Of course, it is not the case when $|p_{z}|$ is

not small. Thus, we see a complete similarity between the case of a particle creation due to a quasiuniform electric field and quasilinear magnetic field for small momenta p_z only. Using (57), we obtain the total number $\mathcal{N}_s$ of created pairs with given s in the form

$$\mathcal{N}_s = \frac{\sqrt{2} - 1 + \ln\left(1 + \sqrt{2}\right)}{16\pi^3} T L_x L_z L_y^2 \left|\mu B'\right|^{5/2} \exp\left(-\frac{\pi m^2}{|\mu B'|}\right). \tag{59}$$

The total number of created pairs with both $s = \pm 1$ is $\mathcal{N} = N_{+1} + N_{-1}$.

The vacuum-to-vacuum transition probability defined in (52) can be calculated in the same way. Then we express it via the total number N as follows

$$P_v = \exp\left(-\beta N\right), \quad \beta = \sum_{l=0}^{\infty} (l+1)^{-3/2} \exp\left(-\frac{l\pi m^2}{|\mu B'|}\right). \tag{60}$$

5 Discussion

It is quite usual to calculate the probability P_v using the Schwinger representation [10]

$$P_v = e^{-2\,\mathrm{Im}\,W},$$

where W is the one-loop effective action. In particular, for the case of of creation of neutral fermions with anomalous magnetic moment this approach was used in [12]. We see that both the total number N of created pairs and $\ln P_v^{-1}$, given by (59) and (60), respectively, are finite for the finite space-time volume of field inhomogeneity and independent of magnetic field strength B_0. In particular, for uniform magnetic field, $B' = 0$, both N and $\ln P_v^{-1}$ are equal to zero. In contrast to the result of Ref. [12], it follows from our results that the arbitrary strong uniform magnetic field is stable with respect of creation of neutral fermions with anomalous magnetic moment and this fact does not depend on the space-time dimensions.

We see that in contrast to the case of a constant electric field, the quantities N and $\ln P_v^{-1}$ are quadratic in L_y. This is a consequence of the fact that the number of states with all possible ω and p_z excited by the field B' is quadratic in the kinetic momentum $|\mu B'|\, L_y$. That is also the reason why the density of created pairs and the density of $\mathrm{Im}\,W$ per unit of length L_y are not constant. In this case the divergence of the effective action W as $L_y \to \infty$ is not linear and it is quite difficult to invent in the framework of the Schwinger approach a reliable method of regularization of W for a linearly growing magnetic field. We believe that this was the main reason of erroneous results in [12]. Note that our approach in the framework of QFT can be used to separate the divergent term of $\mathrm{Im}\,W$ as $L_y \to \infty$ and relate it to pair creation, cf., [27].

It should be noted that the particle creation in the linearly growing magnetic field represents a wide class of physical situations where the magnetic fields have quasilinear heterogeneity in a big enough but restricted areas. One can also see that the leading contribution to differential mean number of created pairs in such fields do not depend on asymptotic behavior of the magnetic field as the size of the heterogeneity tends to the infinity.

Particles with opposite values of spin polarization s have opposite accelerations, the same is valid for antiparticles. The cases $s = +1$ and $s = -1$ differs only by opposite directions of all the motions, and respectively by the opposite dispositions of all the asymptotic ranges. Then, unlike the case of particle creation due to the electric potential step, the neutral particle (antiparticle) created with different s form fluxes aimed in opposite directions. Probabilities of all the processes are equal in both the cases. We see that flux of neutral pair created is formed from fluxes of particle and antiparticle of the same spin polarization and equal intensity. It is the typical property of neutral particle creation by inhomogeneous magnetic field that can be used to observe the effect in astrophysics.

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